Fractal Analysis for Rainfall Characterization: Some Remarks and Possible Future Directions

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Abstract: Applications of the concepts of fractal theory to characterize rainfall have been on the rise everyday. Studies conducted thus far have yielded positive evidence on the fractal nature of rainfall and also revealed the insufficiency of mono-fractal approaches and the necessity of multi-fractal approaches. The assumption behind multi-fractal approaches for rainfall is that the variability of the rainfall process could be directly modeled as a stochastic (or random) turbulent cascade process, since such stochastic cascade processes were found to generically yield multi-fractals. This study investigates the suitability of stochastic multi-fractal approaches (or techniques) for rainfall characterization. The investigation centers on the question whether multi-fractals result only from stochastic (cascade) processes, or could other types of processes also yield multi-fractals. A few commonly used multi-fractal methods are employed to four different types of data sets: (1) artificial stochastic data; (2) artificial chaotic data; (3) real rainfall data from a subtropical climatic region (Leaf River basin, USA); and (4) real rainfall data from an equatorial climatic region (Singapore). The results reveal that the commonly employed multi-fractal techniques might provide positive evidence of multi-fractals not only in stochastic processes but also in chaotic processes, suggesting that the outcomes of the existing multi-fractal techniques must be carefully interpreted. This indicates that a thorough investigation of the dynamical behavior of the rainfall process is necessary to identify the suitable type of fractal approach for rainfall. The possible existence of chaotic behavior in rainfall seems to suggest an alternative, chaotic multi-fractal approach, for rainfall characterization.

Keywords: Rainfall; Fractals; Stochastic; Chaotic; Moment scaling function

1. INTRODUCTION

Recent years have witnessed a large number of studies investigating the presence of fractal behavior in rainfall and the possibility of transformation of rainfall data from one scale to another [e.g. Lovejoy and Mandelbrot, 1985; Schertzer and Lovejoy, 1987; Tessier et al., 1996; Menabde et al., 1997]. Such studies have, on one hand, led to the development and refinement of various techniques to identify if rainfall process exhibits fractal behavior, and, on the other hand, revealed the insufficiency of mono-fractal approaches and the necessity of multi-fractal ones. The multi-fractal approaches originated in turbulence, as turbulence phenomenon was found governed by stochastic (or random) cascade processes [e.g. Mandelbrot, 1974]. The theoretical basis of multi-fractal approaches for rainfall is the assumption that the variability of the rainfall process could be directly modeled as a stochastic turbulent cascade process [e.g. Schertzer and Lovejoy, 1987]. This assumption is supported by the empirical evidence about the fractal properties of rainfall and the analogy with the stochastic cascade models in fully developed turbulence. Even though attempts have been made to justify this assumption, the question whether or not rainfall process is cascade remains unanswered essentially due to the pure phenomenological nature of the stochastic (cascade) approaches [e.g. Menabde et al., 1997].

Having said that, there are at least three possible reasons for the popularity of the stochastic multi-fractal approaches for rainfall. They are: (1) our belief that the seemingly irregular rainfall process is dominantly influenced by a large number of variables; (2) the assumption that the distribution of eddies in the (cascade) rainfall process is stochastic; and (3) the observation of stochastic cascade processes generically yielding multi-fractals. Though the above seem to justify the use of such approaches for rainfall, they are neither necessary nor sufficient, because: (1) rainfall process may be chaotic [e.g. Rodriguez-Iturbe et al., 1989; Sivakumar et al., 1999]; (2) distribution of eddies may follow chaotic behavior [Sivakumar et al., 2001]; and (3) processes that yield multi-fractals may not necessarily be stochastic cascades. In view of the above, the present study attempts to investigate the suitability of stochastic multi-fractal approaches for rainfall characterization by testing whether multi-fractals result only from stochastic (cascade) processes, or could other types of processes, such as chaotic, also yield multi-fractals. This is achieved by employing some of the commonly used multi-fractal techniques to four data sets generated by processes with differing dynamical properties. Among these, two are artificially generated (whose characteristics are known *a priori*): (1) stochastic; and (2) chaotic, and the other two are real rainfall series observed in two different climatic regions (whose characteristics are not known *a priori*): (1) a subtropical climatic region (Leaf River basin, Mississippi, USA); and (2) an equatorial climatic region (Singapore). Three different techniques: (1) the power spectrum; (2) the empirical probability distribution function; and (3) the statistical moment scaling, are employed.

2. MULTI-FRACTAL IDENTIFICATION

Before applying any specific multi-fractal technique to a time series, a common practice is to obtain some information about the general fractal behavior using two standard statistical descriptions: (1) the power spectrum; and (2) the empirical probability distribution function. These two methods are, therefore, employed in the present study. In addition to these, the statistical moment scaling method, one of the methods specifically designed for identifying multi-fractals, is employed. These methods are briefly described next.

2.1 Power Spectrum

The power spectrum, $E(f)$, is useful for studying the oscillations of a signal. In general, the presence of a power-law behavior in the spectrum given by:

$$E(f) \propto f^{-\beta} \tag{1}$$

where f is the frequency and β is the spectral exponent, indicates the absence of characteristic time scale, and thus a multi-fractal behavior.

2.2 Probability Distribution Function

The empirical probability distribution function (*PDF*) of a time series describes the fractal properties of the intensity fluctuations at a given scale, generally the scale corresponding to the measurement resolution. If the series is characterized by a hyperbolic intermittency [e.g. Lovejoy and Mandelbrot, 1985], which may be considered as a feature of (multi-) fractal behavior, then for high intensity threshold values x, the tail of the probability distribution of the series X follows a power law form:

$$Pr(X > x) \propto x^{-q_D} \tag{2}$$

where q_D is the probability exponent. In general, $q_D < 1$ indicates a mono-fractal behavior, whereas a multi-fractal behavior is characterized by a value of $q_D > 1$ [e.g. Tessier et al., 1996].

2.3 Statistical Moment Scaling Function

In the statistical moment scaling method [e.g. Over and Gupta, 1994], the time series is divided into non-overlapping intervals of a certain time resolution. The ratio of the maximum scale of the field to this interval is termed the "scale ratio," λ. For different scale ratios, λ, the average intensity, $\varepsilon(\lambda,i)$, in each interval, i, is computed and raised to power q, and subsequently summed to obtain the statistical moment, $M(\lambda,q)$:

$$M(\lambda,q) = \sum_i \varepsilon(\lambda,i)^q \tag{3}$$

For a scaling field the moment, $M(\lambda,q)$, relates to the scale ratio, λ, as

$$M(\lambda,q) = \lambda^{\tau(q)} \tag{4}$$

where $\tau(q)$ may be regarded as a characteristic function of the fractal behavior. In general, a straight-line behavior in $\tau(q)$ versus q plot is an indication of mono-fractal behavior, whereas a convex function indicates multi-fractal behavior.

3. DATA SETS STUDIED

The tremendous spatial and temporal variability of rainfall is generally believed to be due to the influence of a large number of dominant variables and, therefore, rainfall process is usually treated as a stochastic process. However, it has recently been revealed that the apparently complex rainfall behavior could also be chaotic, i.e. the process is influenced dominantly by only a few nonlinear interdependent variables sensitive to initial conditions [e.g. Rodriguez-Iturbe et al., 1989; Sivakumar et al., 1999 and 2001]. In view of this, in order to achieve the objective stated above, the three multi-fractal techniques are employed first to data sets generated from each of an artificial stochastic and chaotic process, whose characteristics are known *a priori*, and then to two real rainfall data sets from a subtropical climatic region (Leaf River basin, USA) and an equatorial climatic region (Singapore), respectively. The results achieved for the artificial data sets are used as a reference frame to interpret the results achieved for the rainfall data sets. A brief account of these four data sets is presented below.

3.1. Stochastic Data

In general, a stochastic process is one that is influenced by a large number of dominant variables. For such a process, the input parameters are, in general, unknown or only some statistical measures of the parameters are known and, therefore, even short-term predictability is not guaranteed. In this study, the artificial stochastic data set is generated using the random number generation function

$$X_i = rand(\,) \tag{5}$$

3.2. Chaotic Data

A chaotic system is one that looks irregular or erratic but is strictly deterministic and influenced only by a few dominant nonlinear interdependent variables with sensitive dependence on initial conditions. The immediate consequence of sensitive dependence on initial conditions in any system is the impossibility of making perfect predictions or even mediocre predictions sufficiently far into the future. However, the existence of determinism makes it possible to accurately predict the system in the short-term. The artificial chaotic time series generated in the present study is one of the simplest and well-known chaotic series, the Henon map [Henon, 1976]. The Henon map given by:

$$X_{i+1} = a - X_i^2 + bY_i$$
$$Y_{i+1} = X_i \tag{6}$$

yields irregular solutions for many choices of a and b. When a = 1.4 and b = 0.3, a typical sequence of X_i will be chaotic.

3.3 Rainfall Data from a Subtropical Climatic Region (Leaf River basin, Mississippi, USA).

The rainfall data set considered in the present study to represent the subtropical climatic conditions is the daily rainfall series observed over a period of 30 years (January 1963 – December 1992) at the Leaf River basin, in the State of Mississippi, USA. The data represents the mean of the rainfall observed in the basin. The mean annual precipitation in this basin is about 1350 mm and rainfall is generally well distributed throughout the year. March is the wettest month with a mean rainfall of about 160 mm and October is the driest with a mean rainfall of about 80 mm.

3.4 Rainfall Data from an Equatorial Climatic Region (Singapore)

Daily rainfall data observed over a period of 30 years (January 1963 – December 1992) in the island of Singapore is considered to represent the rainfall data from an equatorial climatic region. The mean annual rainfall in Singapore is about 2700 mm. Rainfall in this island occurs during two main rainfall seasons: the Northeast monsoon season from late November to March, and the Southwest monsoon season from late May to September. The Northeast monsoon season is the wetter season accounting for about 48% of the annual rainfall, whereas about 36% of the annual rainfall occurs during the Southwest monsoon season. December is usually the wettest month with an average rainfall of about 280 mm, while July has the lowest average rainfall of about 160 mm. The rainfall data considered in this study represents the mean of the rainfall observed over the island.

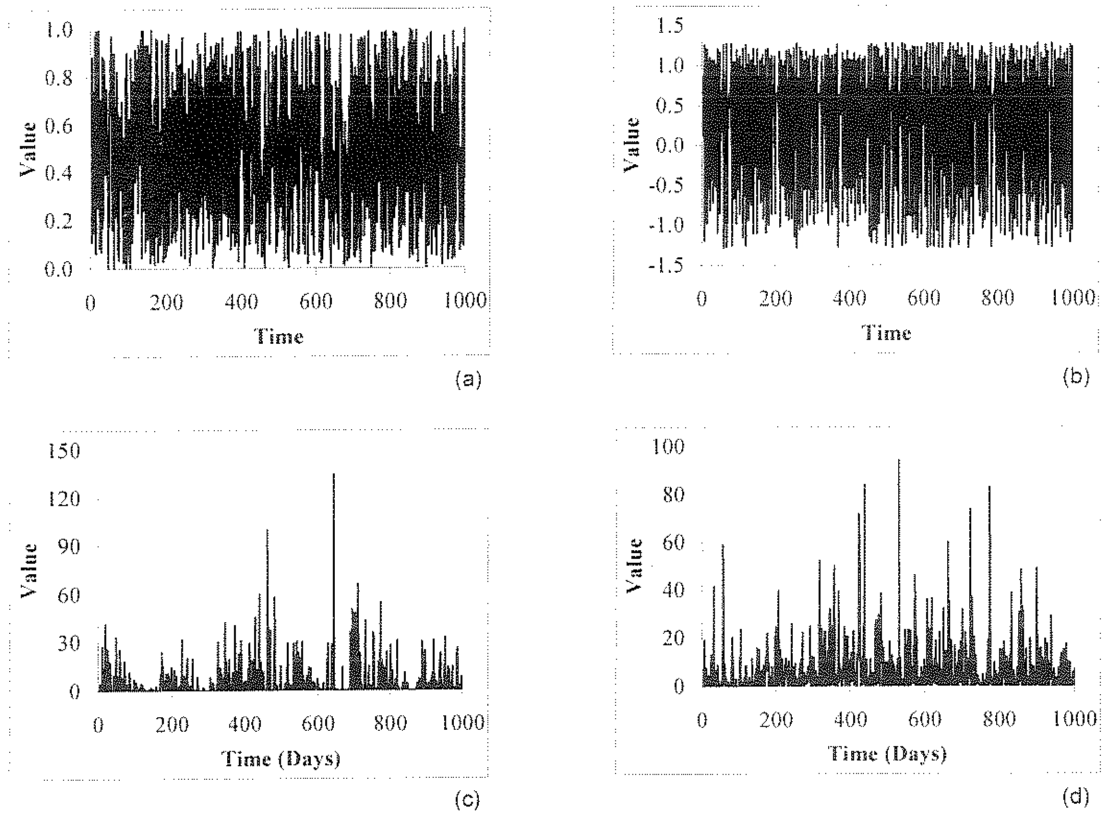

Figure 1. Time Series Plot for (a) Stochastic data; (b) Chaotic data; (c) Leaf River rainfall data; and (d) Singapore rainfall data.

Figures 1(a) to 1(d) show sample time series plots of the above four data sets, respectively, whereas some of the important statistics of these data sets are presented in Table 1.

Table 1. Important Statistics of the four different data sets.

Statistic	Stoch-astic Data	Chao-tic Data	Leaf River Rain-fall Data	Singa-pore Rain-fall Data
No. data	10000	10000	10000	10000
Mean	0.50	0.26	3.86	6.17
St. dev.	0.29	0.72	10.08	11.56
Var.	0.08	0.52	101.7	133.6
Max.	0.99	1.27	221.5	262.8
Min.	0.0002	-1.29	0.0	0.0

4. RESULTS AND DISCUSSION

4.1 Power Spectrum

The spectral exponents, β, obtained for the four data sets analyzed are presented in Table 2. As can be seen, for all the four series β is less than 1, indicating the suitability of an unbounded cascade model. It is interesting to note that the β values obtained for the rainfall series are very close not only to the one obtained for the stochastic series but also for the chaotic series (all of which have $\beta \approx 0.30$). These results indicate that the rainfall process, if found to exhibit multi-fractals, cannot always be considered as stochastic cascade, as chaotic cascade may also be possible.

4.2 Probability Distribution Function

For all the four data sets analyzed in the present study, hyperbolic tail behaviors are observed (Figures not shown). The values of the probability exponents, q_D, obtained for the four series are given in Table 2. The results indicate that: (1) the q_D values obtained for all the four series are above 3.0; and (2) the exponents obtained for the two rainfall series (3.79 and 3.30) are much different and lower than that obtained for the stochastic and chaotic series. Such results imply that: (1) a mono-fractal model is not sufficient for modeling these sets; and (2) the rainfall series may be modeled either/neither as a stochastic or/nor as a chaotic process.

Table 2. Results of fractal analysis for four different data sets.

Statistic	Stochastic Data	Chaotic Data	Leaf River Rainfall Data	Singapore Rainfall Data
β	0.30	0.33	0.30	0.33
q_D	12.5	12.5	3.79	3.30
$\tau(q)$	1.0 (straight line)	0.9-1.1 (irregular curve)	0.9-1.8 (convex curve)	0.9-1.7 (convex curve)

4.3 Statistical Moment Scaling Method

Figures 2(a) to 2(d) respectively show the $\tau(q)$ functions against q for the four data sets. As can be seen, for the stochastic series a straight-line behavior of the $\tau(q)$ function is observed, indicating the presence of mono-fractals. Although the $\tau(q)$ function for the chaotic series looks like those of the stochastic series (i.e. straight-line), this is true only for low values of q. For high values of q, i.e. $q \geq 3.0$, no consistent behavior is observed, such as straight-line or convex. The $\tau(q)$ functions for the two rainfall series are convex curvatures, indicating that the two series exhibit multi-fractals. However, there is no evidence to support the use of stochastic fractal approaches.

5. SUMMARY AND CONCLUSIONS

This study attempted to investigate the suitability of stochastic multi-fractal approaches for rainfall characterization, by testing whether multi-fractals result only from stochastic processes, or even chaotic processes also yield multi-fractals. Three commonly used multi-fractal techniques were employed to four data sets, two artificially generated and two real rainfall series.

The spectral exponents, β, obtained for the two rainfall series were found to be almost the same as that for the stochastic and chaotic series ($\beta \approx$ 0.30) and, therefore, there is no reason to construe that the rainfall processes are only stochastic cascades. Even though, algebraic tails were observed in the probability distribution functions for the two rainfall series, there is no evidence to interpret that they were the outcomes of stochastic cascades because: (1) the probability exponents for the two rainfall series (q_D = 3.79 and 3.30) are significantly different than those for the stochastic series (q_D = 12.5); and (2) even chaotic series yielded the same value (q_D = 12.5) for the exponent as that of the stochastic series. The statistical moment scaling functions, $\tau(q)$, for the two rainfall series were found convex, indicating the possible presence of multi-fractals. However, there is no reason to construe that the two series were the outcomes of stochastic cascades, as the stochastic series did not exhibit a convex behavior in the $\tau(q)$ function, but only a straight-line behavior.

The results from the present study indicated that the commonly employed multi-fractal methods might provide positive (or negative) evidence of the presence of fractals/multi-fractals not only in stochastic processes but also in chaotic processes. These results imply that: (1) the outcomes of the existing stochastic multi-fractal approaches to rainfall must be carefully interpreted; (2) a thorough investigation of the behavior (stochastic or chaotic) of the rainfall process is necessary to identify the suitable type of multi-fractal approach. It is appropriate to note that a recent study by Sivakumar et al. [2001] on the rainfall data observed at the Leaf River basin indicated that the distributions of weights of rainfall data between different resolutions exhibited chaotic behavior. Also, studies have reported convincing evidence regarding the presence of chaotic behavior in Singapore rainfall data [e.g. Sivakumar et al., 1999]. A possible implication of these may be that the chaotic approach can also be a suitable framework for rainfall characterization from a scale-invariance point of view.

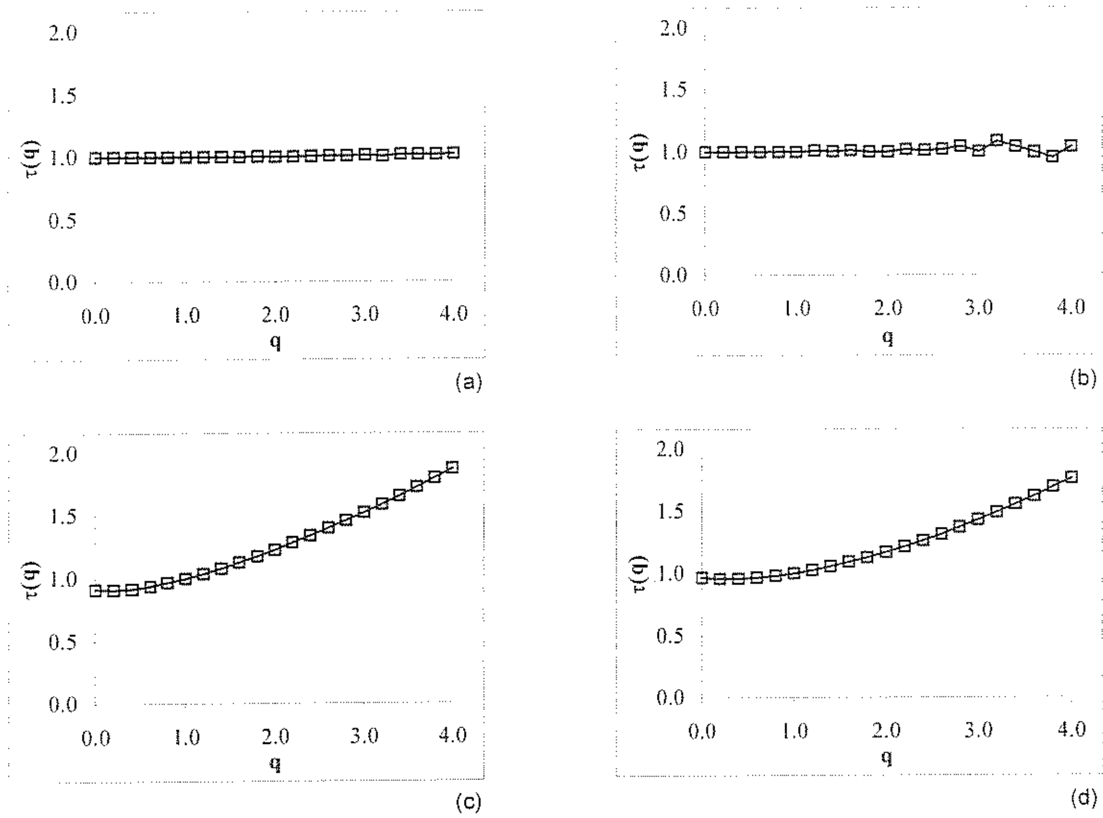

Figure 2. Statistical Moment Scaling Function for: (a) Stochastic data; (b) Chaotic data; (c) Leaf River rainfall data; and (d) Singapore rainfall data.

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